Upper Paraguay river at Porto Murtinho, Mato Grosso do Sul, Brazil, featuring a flood hydrograph lasting one year (the maximum possible), clearly the quintessential flood wave.

WHEN IS THE DIFFUSION WAVE APPLICABLE?

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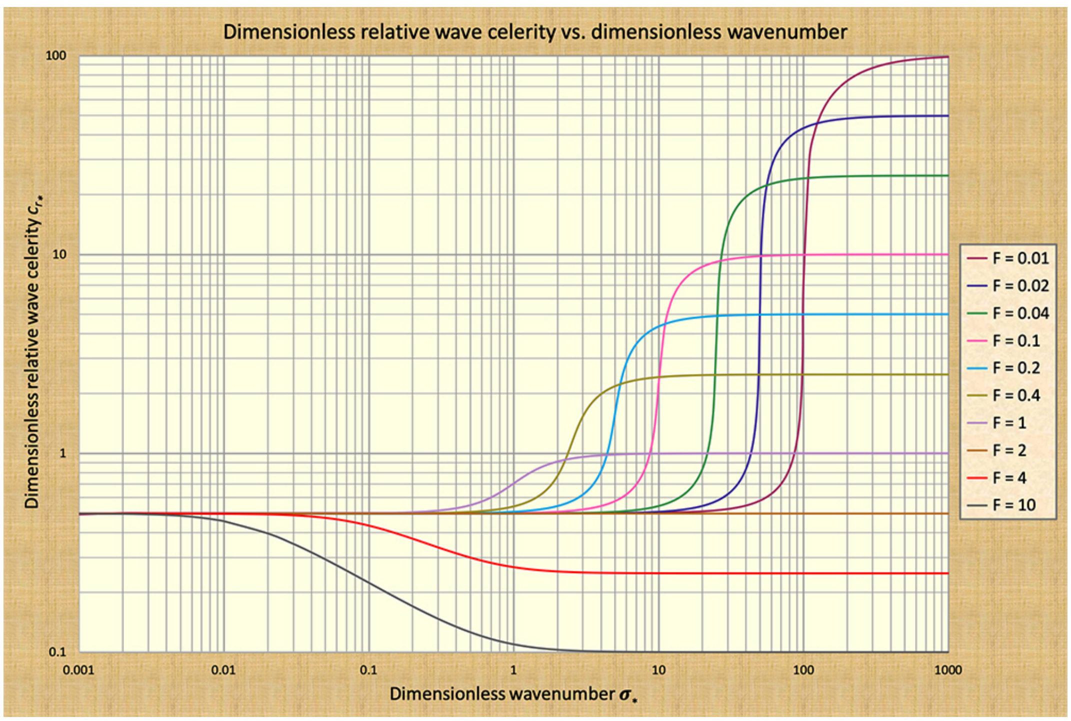
ABSTRACT. A review of diffusion waves and their use for routing flood waves has been accomplished. Diffusion waves travel with the Seddon celerity, i.e., the kinematic wave celerity, and are subject to little attenuation (diffusion). These properties distinctly match those of typical flood waves. Other free-surface flow waves, namely, kinematic, mixed, and dynamic, are either nondiffusive (kinematic and dynamic), or too diffusive (mixed). In particular, the mixed waves are confirmed to be so greatly diffusive as to question their mere existence altogether. Criteria for the applicability of both kinematic and diffusion waves show that the latter, the diffusion waves, have a broader range of applicability than do the kinematic waves. Therefore, the diffusion wave is recommended for practical applications in flood hydrology.

1. INTRODUCTION

The diffusion wave is a type of wave used in flood routing. Other types of waves are the kinematic wave and the mixed kinematic-diffusion wave, hereafter called "mixed wave" (**Lighthill and Whitham, 1955**; **Ponce and Simons, 1977**). Note that the mixed wave has been widely referred to in the literature as "dynamic wave", although this usage appears to be ill-advised, because it leads to a semantic confusion with the long-established dynamic wave of **Lagrange (1788)**, a concept quite different from the mixed wave.

A comprehensive classification of shallow-water waves in open-channel flow was accomplished by **Ponce and Simons (1977)**, who used linear stability theory to derive the celerity and attenuation functions of *all four* types of shallow-water waves: (1) kinematic waves, (2) diffusion waves, (3) mixed waves, and (4) dynamic waves. These wave types are defined in terms of the dimensionless wavenumber σ_*, as shown in Fig. 1 (**Ponce, 2023**). Kinematic waves correspond to the smallest values of σ_* (to the left of the scale), and dynamic waves to the largest (to the right of the scale). Mixed waves lie along the middle-to-right of the scale, while diffusion waves lie along the left-of-middle.



Ponce and Simons (1977)

Fig. 1 Dimensionless relative wave celerity c_{r_*} vs dimensionless wavenumber σ_*.

In Figure 1, kinematic waves lie along the first log cycle (to the left), while dynamic waves lie along the fifth and sixth cycles (to the right), depending on the Froude number. Mixed waves lie mostly along the fourth and fifth cycles, while diffusion waves lie along the second and third cycles.

Hydrodynamic theory asserts that if the wave celerity is a constant along the dimensionless wavenumber, *wave attenuation is zero*. In Figure 1, this condition corresponds to both kinematic waves (left of scale) and dynamic waves (right of scale). Conversely, if the wave celerity varies along the dimensionless wavenumber, as in the case of a mixed wave, *wave attenuation is nonzero*. Wave attenuation reaches a *maximum* at the point of zero curvature of the dimensionless relative celerity function c_{r*} vs dimensionless wavenumber σ_* (Fig. 1), i.e., when the second derivative is equal to zero. Experience indicates that these maximum values of wave attenuation may actually render the wave in question nonexistent, due to the extremely high rates of attenuation (**Lighthill and Whitham, 1955**).

Thus, the question remains: If the kinematic waves have no attenuation, and the mixed waves are subject to very strong attenuation, what happens to the diffusion waves, which ostensibly lie in between them? The answer is: The diffusion waves are subject to a small but *finite* amount of attenuation, which is generally much smaller than the strong attenuation featured by mixed waves.

In this article, we make the case for the diffusion wave. We note that if the flood wave has a small amount of attenuation, the diffusion wave model will account for it, while the kinematic wave model will not. Furthermore, we show that due to the large amounts of attenuation which are predicted for mixed waves, the latter are not very likely to occur in the real world. In Section 2, we explain the nature of flood waves and make a point of the need to focus on the diffusion wave. With the applicability question clearly answered in Section 4, the time has come to hail the diffusion wave as the method of choice in flood routing engineering practice.

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2. NATURE OF A FLOOD WAVE

What is the nature of a flood wave? Essentially, a flood wave is a "long" wave, i.e., one of small dimensionless wavenumber (Fig. 1), traveling at, or very close to, the kinematic wave celerity, and experiencing little attenuation. **Seddon (1900)** pioneered the study of flood waves, concluding that its celerity could be expressed as the slope of the discharge-area rating $Q = \alpha A^{\beta}$, in which Q = discharge, A = flow area, and α and β are coefficient and exponent, respectively.

According to Seddon, the celerity of a flood wave is: $c = dQ/dA$, in which $dA = (1/T)dy$, with T = (stream) channel top width, and y = stage, or water surface elevation. He expressed the celerity of a flood wave as $c = (1/T)\ dQ/dy$. Thus, a flood wave is essentially a kinematic wave subject to a relatively small amount of diffusion. Indeed, this is the kinematic-wave-with-diffusion of **Lighthill and Whitham (1955)** or, more concisely, the diffusion wave of **Ponce and Simons (1977)**.

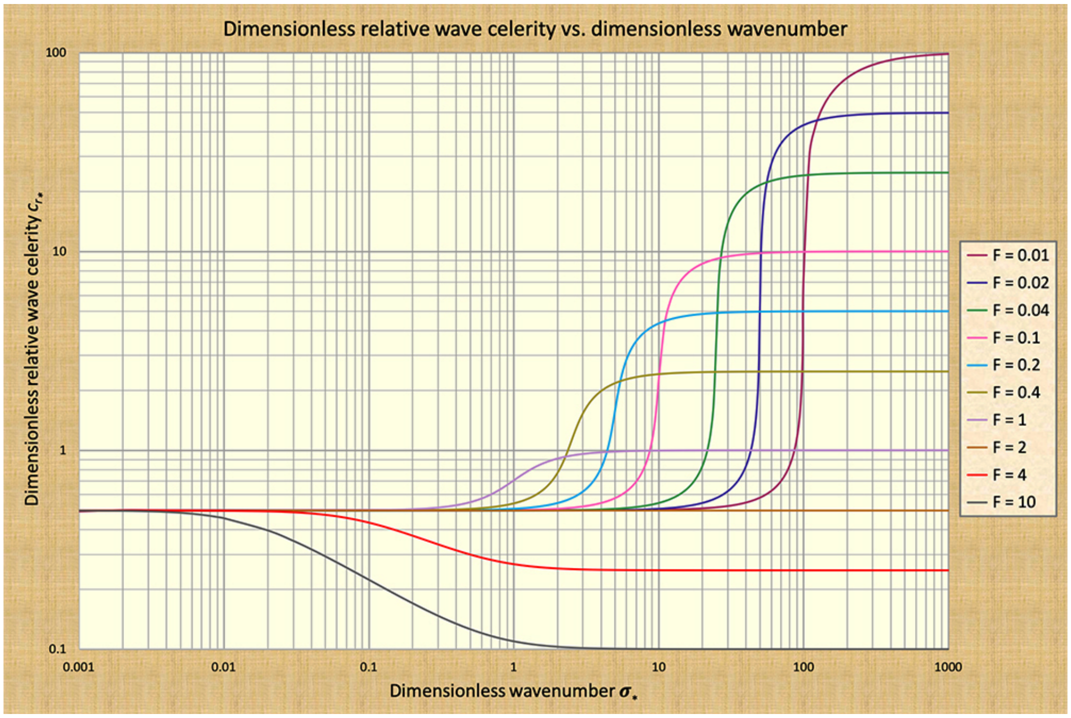
As long as diffusion needs to be accounted for, diffusion waves may not be modeled with kinematic waves, because the latter feature *zero* diffusion. We note that in the 1980s, kinematic waves were solved using numerical models, and the latter *did feature* some diffusion. This diffusion, however, was uncontrolled *numerical diffusion*, and not related to the true physical diffusion of the flood wave; therefore, the results of the routing varied with the choice of grid size.

Could the flood wave be construed as a mixed wave, a wave that sits on the middle-to-right of the dimensionless wavenumber spectrum (Fig. 1)? The answer is: Not likely for typical flood waves, which hold their stage and do not attenuate very much. If the flood wave were to attenuate strongly, it would cease to be a flood wave, instead joining the mass of the underlying equilibrium, steady flow. Thus, we conclude that mixed waves are not an appropriate model of flood waves, at least, not in the general case. In the following section, calculations of diffusion waves will confirm these statements.

Having placed aside: (a) the kinematic waves, because they lack diffusion entirely, and (b) the mixed waves, because they have too much diffusion, we are left only with the diffusion wave, which lies in between kinematic and mixed waves in the dimensionless wavenumber spectrum. This is the wave that truly embodies the nature of flood waves: A kinematic wave featuring a small, but perceptible, amount of diffusion.

3. THE DIFFUSION WAVE

According to theory, the dimensionless relative celerity of a diffusion wave resembles that of a kinematic wave, but unlike the latter, it increases, ever so slightly with the dimensionless wavenumber σ_* (Fig. 1). This increase is the source of the wave diffusion which characterizes the diffusion wave.



Ponce and Simons (1977)

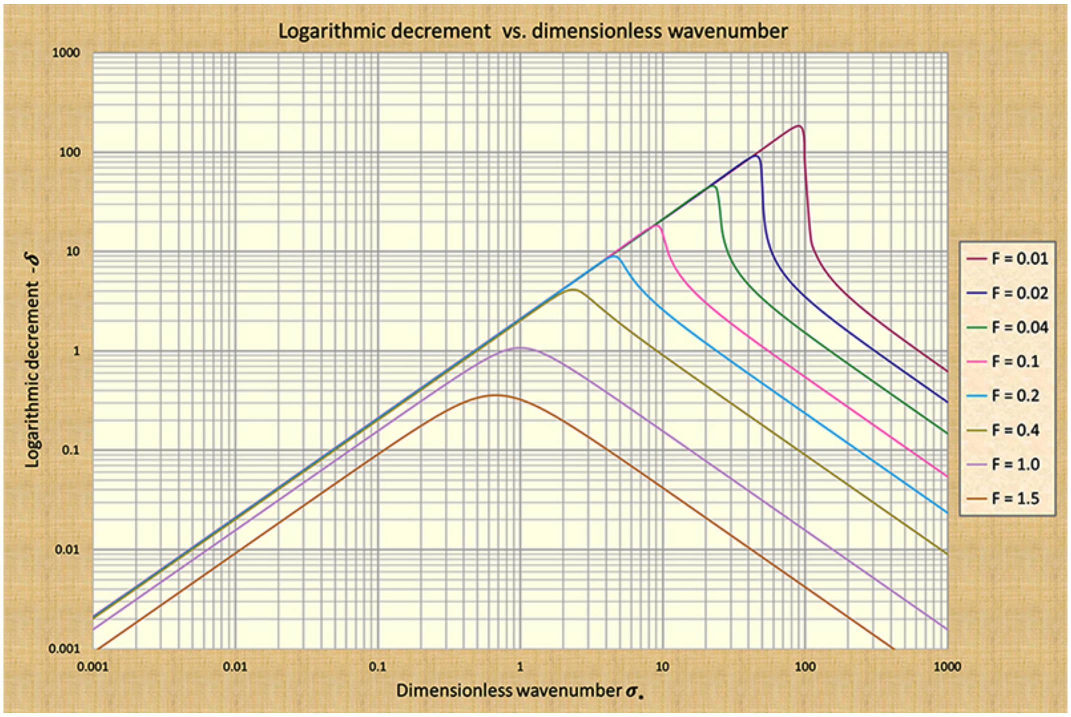
Fig. 1 Dimensionless relative wave celerity c_{r*} vs dimensionless wavenumber σ_*.

The amount of wave diffusion is expressed in terms of the *logarithmic decrement* δ, a measure of the rate at which the wave changes upon propagation (**Wylie, 1966**). The definition of logarithmic decrement is: $\delta = \ln Q_1 - \ln Q_0$, or, alternatively: $Q_1 = Q_0\ e^{\delta}$, in which Q_0 = flood discharge at the start of the measurement, and Q_1 = flood discharge after an elapsed time equal to one (sinusoidal) period of propagation.

The discharge decreases for a negative value of δ, causing wave attenuation, corresponding to Froude number **F** < 2 (Vedernikov number **V** < 1); it increases for a positive value, a *logarithmic increment*, causing wave amplification, corresponding to Froude number **F** > 2 (**V** > 1) (**Ponce, 1991**).

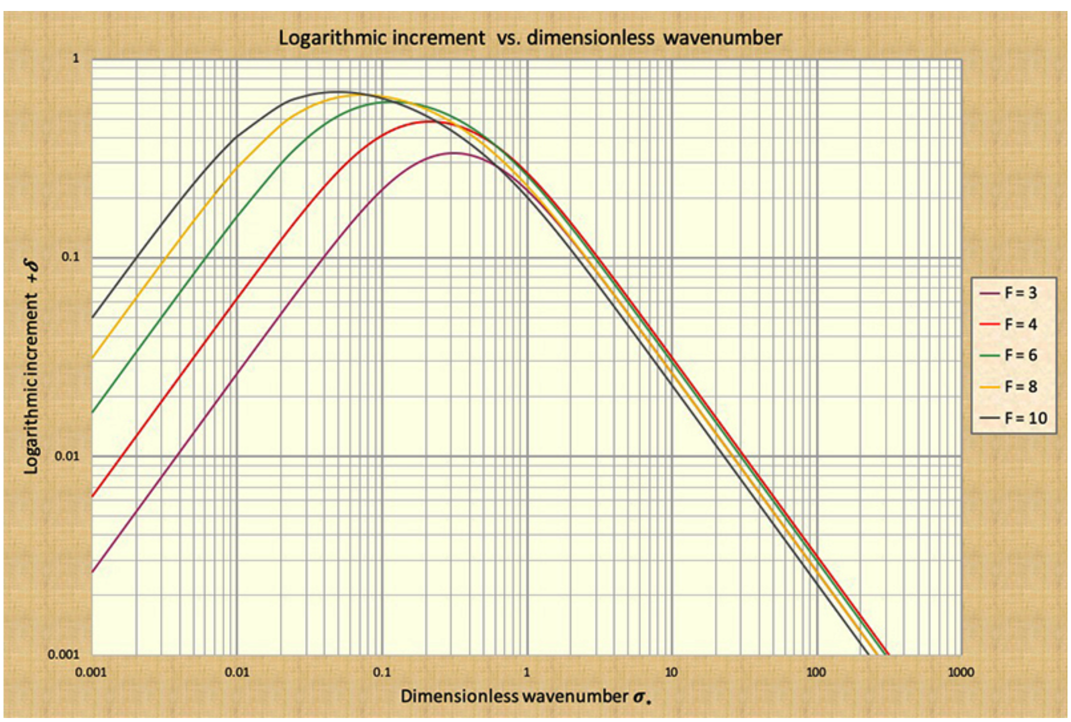
Ponce and Simons (1977) have used linear stability theory to calculate the logarithmic decrement of the diffusion wave. The expression is: $\delta_d = (2\ \pi /3)\ \sigma_*$. Note that in this expression, as $\sigma_* \rightarrow 0$, the logarithmic decrement $\delta_d \rightarrow 0$, confirming that a kinematic wave is not subject to attenuation. Moreover, note that for $\sigma_* \rightarrow \infty$, the diffusion wave logarithmic decrement $\delta_d \rightarrow \infty$, confirming the inability of the diffusion wave logarithmic decrement formula to account for the dynamic waves, which also feature zero attenuation.

Figure 2 shows the variation of the logarithmic decrement for *all* wave types, through the range of dimensionless wavenumbers from 0.001 to 1000, for Froude numbers **F** < 2. Moreover, Figure 3 shows the variation of the logarithmic *increment* for *all* wave types, through the range of dimensionless wavenumbers from 0.001 to 1000, for Froude numbers **F** > 2.



Ponce and Simons (1977)

Fig. 2 Logarithmic decrement $-\delta$ vs dimensionless wavenumber σ_* for Froude **F** < 2.



Ponce and Simons (1977)

Fig. 3 Logarithmic increment $+\delta$ vs dimensionless wavenumber σ_* for Froude **F** > 2.

Table 1 shows values of the diffusion wave logarithmic decrement δ_d relevant in the present context. The examination of this table leads to the conclusions summarized in **Box A.**

Table 1. Logarithmic decrement of the diffusion wave across the dimensionless wavenumber spectrum.

[1]	[2]	[3]	[4]	[5]	[6]
No.	**Dimensionless wavenumber σ_***	**Logarithmic decrement δ_d**	**e^{δ_d}**	**Wave attenuation ($1 - e^{\delta_d}$)**	**Wave type**
0	0.0001	0.00020944	1	0	Kinematic
1	0.001	0.0020944	0.998	0.002	Kinematic/ diffusion
2	0.01	0.020944	0.979	0.021	Diffusion
3	0.1	0.20944	0.811	0.189	Diffusion
3a	0.17	0.35604	0.700	0.300	Diffusion/ Mixed
4	1	2.0944	0.123	0.877	Mixed
5	10.	20.944	0	1	Mixed
6	100.	N.A.	N.A.	N.A.	Dynamic [1]
7	1000.	N.A.	N.A.	N.A.	Dynamic [1]

[1] In Col. 6, lines 6 and 7 are labeled as "Dynamic". The diffusion wave logarithmic decrement δ_d does not apply after the peak of attenuation is reached, i.e., in the dynamic wave range (see Fig. 2).

Box A. Conclusions from Table 1.

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1. For very small dimensionless wavenumbers (see Line 0), the wave attenuation, shown in Col. 5, is zero, i.e., a kinematic wave.
2. For small dimensionless wavenumbers (see Line 2), the wave attenuation, shown in Col. 5, is 0.021 (2.1%) , i.e., a diffusion wave.
3. Wave attenuation reaches 0.3, a high number (30%) for σ_* = 0.17; i.e., at the limit between diffusion and mixed waves.
4. Wave attenuation reaches 0.877, a very high number (87.7%), for σ_* = 1 , i.e., a mixed wave.
5. Wave attenuation reaches 1, the maximum value (100%), for σ_* = 10, i.e., a mixed wave.

We confirm that kinematic waves are not subject to attenuation. We also confirm that for the midrange value of dimensionless wavenumber $\sigma_* > 0.17$, the wave attenuation is greater than 30%, a threshold which is widely considered to be the division between diffusion waves (of limited wave diffusion, less than 30%) and mixed waves (of unlimited wave diffusion, which could readily reach 100%) (**Flood Studies Report, 1975**). Thus, the rationale for the argument that mixed waves are very strongly dissipative and, in most cases of practical interest, that they will not likely be there for us to calculate them (**Ponce, 1992**).

We confirm the following observations, based on detailed calculations, in reference to Fig. 1: (a) kinematic waves lie along the first log cycle; (b) diffusion waves along the second and third, (c) mixed waves along the fourth and fifth, and (d) dynamic waves along the fifth and sixth cycles, the latter depending heavily on the Froude number. Kinematic waves do not attenuate, mixed waves attenuate very strongly, and dynamic waves, being characteristically short, do not resemble flood waves, which are demonstrably long. Thus, a strong case is made for the applicability of the diffusion wave for flood routing applications.

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4. APPLICABILITY OF DIFFUSION WAVES

Having shown that diffusion waves are applicable to the flood routing problem, we now show just how applicable they are. Here we elaborate on the work of **Ponce *and others* (1978)**, who established the criterion for the applicability of kinematic and diffusion waves in terms of flood wave properties.

For the kinematic wave model, **Ponce *and others*** stated that to achieve at least 95% accuracy of the kinematic wave solution *after one period of propagation*, the dimensionless period has to satisfy the following inequality: $\tau_* \geq 171$. The dimensionless period is defined as follows: $\tau_* = T\ S_o\ (u_o\ /d_o)$, in which T = period of the perturbation, S_o = channel (stream) bottom slope, u_o = mean equilibrium flow velocity, and d_o = equilibrium flow depth. For example, given the following data: u_o = 3 fps, d_o = 10 ft, and S_o = 0.0001, the resulting wave period is: $T > (171 \times 10)\ /\ (\ (0.0001 \times 3)$ = 5,700,000 seconds = 65.97 days. In other words, the duration of the flood must be greater than about 66 days for the kinematic wave solution to be at least 95% accurate after one period or propagation. It follows that the longer the flood duration, the more kinematic the flood wave is, and this finding agrees admirably with the theory.

For the diffusion wave model, **Ponce *and others*** compared the logarithmic decrement of the diffusion wave δ_d with that of the full solution (Fig. 2), and concluded that to achieve at least 95% accuracy of the diffusion wave solution *after one period of propagation*, the dimensionless period has to satisfy the following inequality: $\tau_* \geq 30$. In this case, the dimensionless period is defined as follows: $\tau_* = T\ S_o\ (g\ /d_o)^{1/2}$, in which g = gravitational acceleration. Using the previous example for comparison, for d_o = 10 ft, and S_o = 0.0001, the resulting wave period is: $T > 30\ /\ (\ (0.0001 \times [32.17/10)^{1/2}]$ = 167,261 seconds = 1.93 days. In other words, the duration of the flood wave must be greater than 1.93 days (about 2 days) for the diffusion wave solution to be at least 95% accurate after one period or propagation. This finding reveals that the theory of diffusion waves is correct: These waves will apply for a larger number of cases than the kinematic wave. For the example presented here, a flood wave would need to have a minimum duration of 66 days for the kinematic wave to apply; however, for the diffusion wave, a minimum of only 2 days will suffice. Thus, we confirm the broad range of practical cases for which the diffusion wave is applicable.

5. APPLICATION TO FIELD DATA

The concepts elaborated in the previous section are applied here to the Upper Paraguay river, in Mato Grosso do Sul, Brazil, and neighboring Bolivia. This river features a unique geographical setting, flowing through a continental delta encompassing the Pantanal of Mato Grosso, the largest wetland in the world, spanning 136,700 km^2. The geomorphological and hydrological setting of the Upper Paraguay river has been described by **Ponce (1995)**.

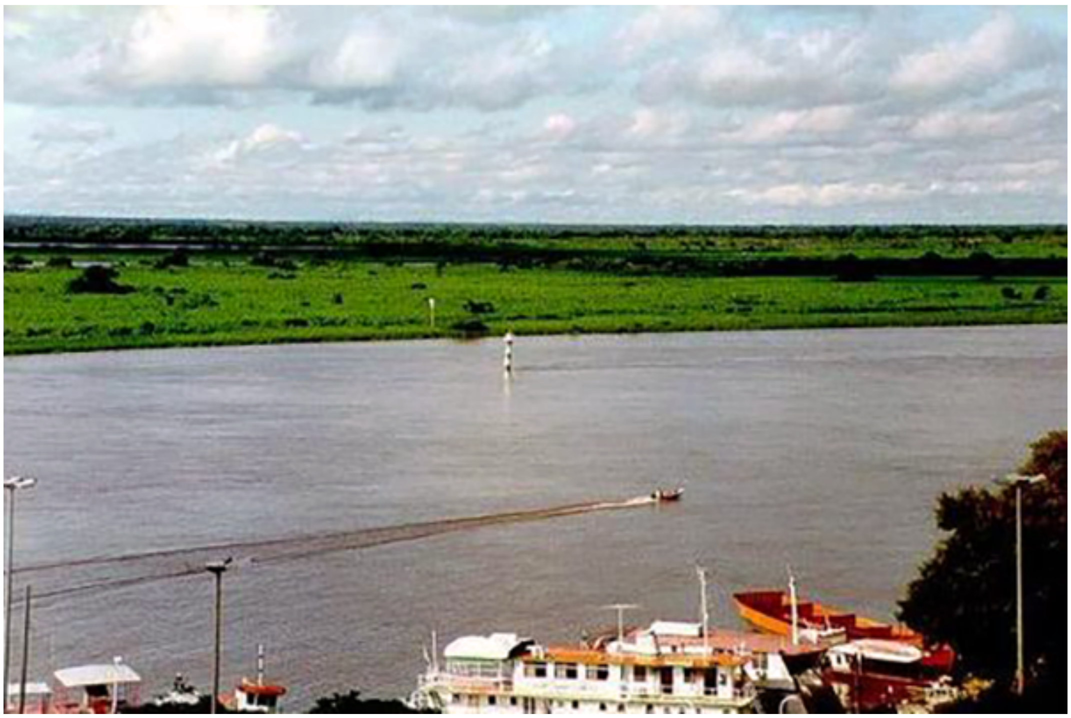

Fig. 4 Upper Paraguay river near Ladario, Mato Grosso do Sul, Brazil.

The flood hydrograph of the Upper Paraguay river, in its lower reaches, from Ladario to Porto Murtinho, comprising a distance of 520 km along the river, features *only one peak flow stage*, a condition attributed to the extreme runoff diffusion caused by the very small slope of the river channel, which varies between 2.34 cm/km near Ladario, and 0.83 cm/km near Porto Murtinho, for an average of 1.6 cm/km.

Downstream of Ladario, the river stage rises from March to August, receding from September to February. The peak stage occurs typically in the month of June and the low stages in December. The mean flow depth varies between 5.24 m at Ladario and 12.28 m at Porto Murtinho, for an average of 8.76 m. Moreover, the mean speed of propagation of the flood wave has been estimated at 0.1 m/s (**Ponce, 1995**). Thus, the values relevant to the assessment of flood wave model applicability are the following: T = 12 months, S_o = 0.000016, u_o = 0.1 m/s, and d_o = 8.76 m. The calculations are summarized in **Box B**.

Fig. 5 Upper Paraguay river near Porto Murtinho, Mato Grosso do Sul, Brazil.

Box B. Summary of calculations of flood wave model applicability.

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1. Relevant data: T = 12 months, S_o = 0.000016, u_o = 0.1 m/s, and d_o = 8.76 m.

2. Kinematic wave applicability relation: $\tau_* = T\, S_o\, (u_o / d_o) \geq 171$

3. τ_* = 12 mo × 30 days/mo × 86,400 sec/day × 0.000016 × 0.1 m/sec / 8.76 m = 5.68

4. τ_* = 5.68 << 171. Therefore, the kinematic wave model **is not applicable.**

   This is attributed to the very mild channel slope (1.6 cm/km).

5. Diffusion wave applicability relation: $\tau_* = T\, S_o\, (g / d_o)^{1/2} \geq 30$.

6. τ_* = 12 mo × 30 days/mo × 86,400 sec/day × 0.000016 × (9.81 m/sec^2 / 8.76 m)$^{1/2}$ = 527

7. τ_* = 527 >> 30. Therefore, the diffusion wave model **is applicable.**

   Even with this very small channel slope (1.6 cm/km), the diffusion wave model remains applicable.

6. CONCLUDING REMARKS

A review of diffusion waves and their use for routing flood waves has been accomplished. Diffusion waves travel with the Seddon celerity, i.e., the kinematic wave celerity, and are subject to little attenuation (diffusion). These properties distinctly match those of typical flood waves. Other free-

surface flow waves, namely, kinematic, mixed, and dynamic, are either nondiffusive (kinematic and dynamic), or too diffusive (mixed). In particular, the mixed waves are confirmed to be so greatly diffusive as to question their mere existence altogether. Criteria for the applicability of both kinematic and diffusion waves show that the latter, the diffusion waves, have a broader range of applicability than do the kinematic waves. Therefore, the diffusion wave is recommended for practical applications in flood hydrology. An application to field data from the Upper Paraguay river, in Mato Grosso do Sul, Brazil, further confirms the findings of this study.

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