A tributary of the La Leche river, Lambayeque, Peru.

WHY IS THE CASCADE OF LINEAR RESERVOIRS A METHOD OF CHOICE IN UNIT HYDROGRAPH ANALYSIS?

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ABSTRACT. A review of the method of cascade of linear reservoirs for unit hydrograph development is presented, explained, and clarified. Section 1 compares the cascade with the synthetic unit hydrographs of Snyder and NRCS. Section 2 presents basin modeling concepts, including Hayami's diffusivity, which characterizes the diffusion of one-dimensional free-surface flows. Section 3 reviews concepts of unit hydrograph theory. Section 4 explains the methodology of cascade of linear reservoirs, deriving the routing equation and displaying actual online calculations. Section 5 provides an analysis of the cascade, focusing on its capability to model a broad range of runoff diffusion effects. Section 6 describes the geomorphological approach to estimate the parameters of the cascade. Section 7 provides several online calculations, including an actual practical application. In summary, the cascade of linear reservoirs is predicated on its capability to model a broad range of diffusion effects, enabling increased accuracy for simulating synthetic unit hydrographs. The online computational capability enhances the method's utility for the effective modeling of runoff diffusion to solve a wide variety of flooding problems.

1. INTRODUCTION

A major aim of hydrologic analysis is the calculation of the flood hydrograph and the associated peak discharge for a given flood frequency. For small catchments, say less than 1 square mile (2.5 km^2), the peak discharge is all that may be required. In this case, the rational method is generally the method of choice (**Ponce, 2014a: Rational**). For larger basins, typically of medium size, say between 1 and 400 square miles (1,000 km^2), it is necessary to account for *runoff diffusion* and the shape of the flood hydrograph in a more precise way than the rational method is able to do (**Ponce, 2014a: Diffusion**). The recognition of this fact led **Sherman (1932)** to develop the *unit hydrograph*, i.e., a hydrograph corresponding to a *unit depth* of runoff, either 1 inch or 1 cm. In order to obtain the composite flood hydrograph, Sherman *convoluted* the unit hydrograph with the *effective storm hyetograph* in a procedure referred to as *unit hydrograph convolution* (**Ponce, 2014a: Convolution**).

In the decades that have elapsed since the time of its original development, the concept of unit hydrograph has undergone major improvements in both theory and practice. Sherman's pioneering work was followed shortly thereafter by the first synthetic unit-graph (Snyder, 1938; **Ponce, 2014a: Snyder**), and later by the NRCS's synthetic unit hydrograph (Natural Resources Conservation Service, 1985; **Ponce, 2014a: NRCS**). The methodology known as the *cascade of linear reservoirs* (CLR) was developed in the 1950s as the routing component of the *Streamflow Synthesis and Reservoir Regulation Model* (SSARR), originally applied to the Columbia River Basin (U.S. Army Crops of Engineers, 1975; **Ponce, 2014a: Cascade**).

This article focuses on the cascade of linear reservoirs as a method of choice in unit hydrograph analysis. The method is shown to have broad, almost unlimited flexibility in simulating *any and all* amounts of runoff diffusion; thus, being singularly qualified to solve the unit hydrograph and the associated composite flood hydrograph problem. The cascade of linear reservoirs method is presented here as the *only* synthetic unit hydrograph methodology capable of simulating the broad range of runoff diffusion problems that are likely to be encountered in actual practice.

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2. MODELING CATCHMENTS/WATERSHEDS/BASINS

The modeling of surface flow in catchments, watersheds, and basins, hereafter referred to as "basins", is no simple matter. The objective is to convert (transform) effective rainfall into surface runoff. Effective rainfall is the amount of rainfall (in inches or centimeters) after the hydrologic abstractions (infiltration, evapotranspiration, surface storage, etc.) have been subtracted. There are two general ways to accomplish this aim: (1) *lumped* method, and (2) *distributed* method (**Ponce, 1991**). The lumped method will provide *only one* hydrograph of surface runoff, at the mouth of the basin, its farthest downstream point. On the other hand, the distributed method will provide a hydrograph *at any point* inside the basin, regardless of its location with respect to the basin's mouth (Fig. 1). A lumped approach relies on the unit hydrograph, while a distributed approach is based on a set of physically based governing equations and associated numerical models.

Fig. 1 Little Butte Creek watershed, Jackson County, Southwest Oregon.

The overriding objective of surface-runoff mathematical modeling is to accurately describe the relevant physical processes of *convection* and *diffusion*. Convection, a first-order process, is akin to the movement of the parcels of water in a direction *parallel* to the underlying terrain. Diffusion, a second-order process, describes the movement of the parcels in a direction *perpendicular* to the terrain. The combination of these two processes transforms a *hyetograph*, a histogram of effective rainfall, defined in terms of a series of depths (cm) and an interval duration (hr), into a *hydrograph*. The latter is a plot of the variation of discharge in time, with clearly defined characteristics such as peak discharge and time-to-peak. Being of first order, convection is the primary, or greater process; diffusion, the second-order process, is normally small and often negligible. In certain cases, however, diffusion may become significant enough to actually affect or interfere with the amount of convection. The accurate determination of the diffusion component is at the crux of surface-runoff modeling, constituting the central topic of this article.

Runoff diffusion is the component of surface runoff that is produced by diffusion. The amount of diffusion was first quantified by **Hayami (1951)**, who combined the governing equations of water continuity and motion, i.e., the Saint-Venant equations (Saint Venant, 1849; **Ponce, 2014b: Governing**), into a second-order partial differential equation with discharge as the dependent variable. Hayami expressed the diffusion coefficient ν, or Hayami's diffusivity, as follows:

$$\nu = \frac{q_o}{2 S_o} \qquad (1)$$

in which q_o = unit-width discharge, and S_o = bottom slope. It is seen that runoff diffusion is directly proportional to the unit-width discharge and inversely proportional to the prevailing bottom slope.

The unit-width discharge q_o varies within a narrow range, typically less than two orders of magnitude (10-40 m^2/s, or alternatively 1-100 m^2/s) (**Ponce and others, 2003**). However, the bottom slope S_o

varies within a much broader range, typically from steeper than 0.1 to milder than 0.00001, i.e., at least five orders of magnitude (**Ponce, 1995**). Therefore, the bottom slope is seen to be the controlling factor in assessing the amount of runoff diffusion. For bottom slopes greater than 0.01, diffusion is negligible, and an unsteady flow feature may be construed as a *kinematic wave*, featuring only convection. On the other hand, for bottom slopes smaller than 0.001, diffusion may be substantial, and the unsteady flow feature is essentially diffusive, properly a *diffusion wave* (**Ponce, 2023a**). Eventually, as the bottom slope approaches zero, i.e., in the presence of flat land, diffusion tends to infinity, resulting in ponding, with the total absence of surface runoff. At that point, surface-water evaporation takes over.

3. UNIT HYDROGRAPH THEORY

The rational method has no effective way of accounting for runoff diffusion. As basins increase in surface area from small (A < 2.5 km^2) to midsize (2.5 ≤ A < 1,000 km^2), runoff diffusion becomes increasingly important and, therefore, a more precise calculation may be required. The larger the basin area, the more likely it is that the mean terrain slope will be milder, although in certain unusual cases this may not be necessarily so. In general, the larger the basin, the greater the amount of runoff diffusion; conversely, the smaller the basin, the smaller the amount of runoff diffusion.

To develop a way to account for runoff diffusion, **Sherman (1932)** envisioned that he would *typify* the runoff diffusion of a given basin in terms of a **unit hydrograph**, defined as the hydrograph corresponding to a unit depth of runoff (1 cm or 1 in) lasting "a unit increment" of time [Fig. 2 (a)]. In Sherman's concept of unit hydrograph, the word *unit* was meant to refer not to the "unit depth" of runoff, but rather to the fact that the unit runoff was *indivisible*. Thus, a basin could have several 1-cm unit hydrographs, for instance: (a) the 1.0-cm/hr 1-hr unit hydrograph, (b) the 0.5-cm/hr 2-hr, (c) the 0.25-cm/hr 4-hr, and so on, all of these hydrographs representing 1 cm of runoff. [In essence, once the unit hydrograph duration is chosen, it may not be split]. After one of these unit hydrographs is identified either through field measurement or by a synthetic calculation, the others may be determined based on the first one (**Ponce, 2014a: Change in UH Duration**).

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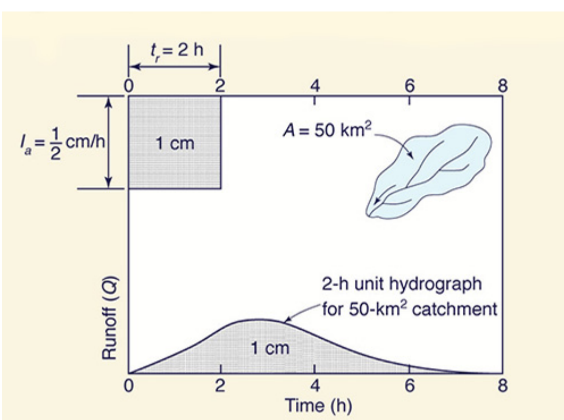

Fig. 2 (a) The unit hydrograph.

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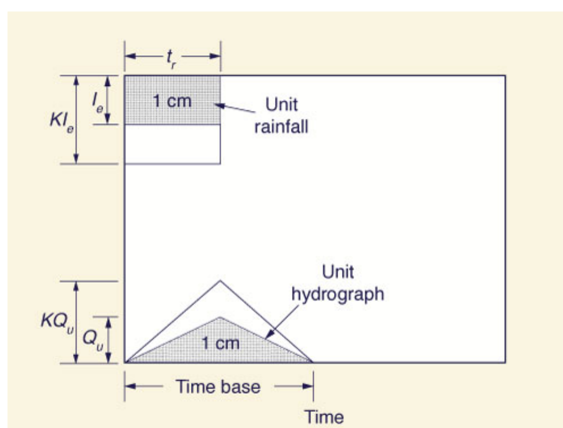

Fig. 2 (b) The unit hydrograph: Linearity.

Once the unit hydrograph had been determined for a given basin, Sherman proceeded to apply the convolution technique to calculate the composite flood hydrograph for the corresponding effective rainfall hyetograph (the effective storm pattern). The procedure hinges on an assumption of linearity, that is, given a multiplier K, the hydrograph response to rainfall amount KI_e is taken as KQ_e, in which Q_e = unit hydrograph peak [Fig. 2 (b)]. Given the effective storm hyetograph, the lagging in time of the several hydrograph increments [Fig. 2 (c)] results in the composite flood hydrograph [Fig. 2 (d)] (**Ponce, 2014a: Composite**).

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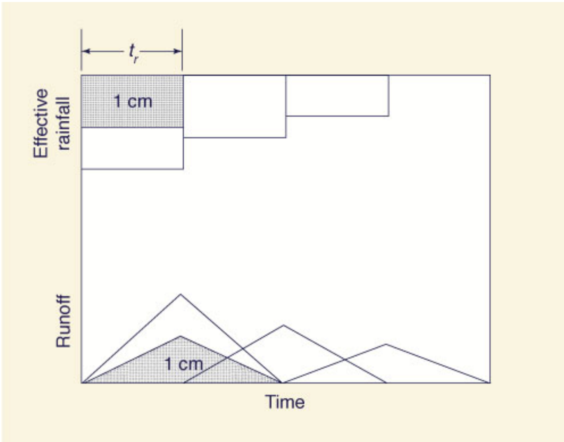

Fig. 2 (c) The unit hydrograph: Lagging.

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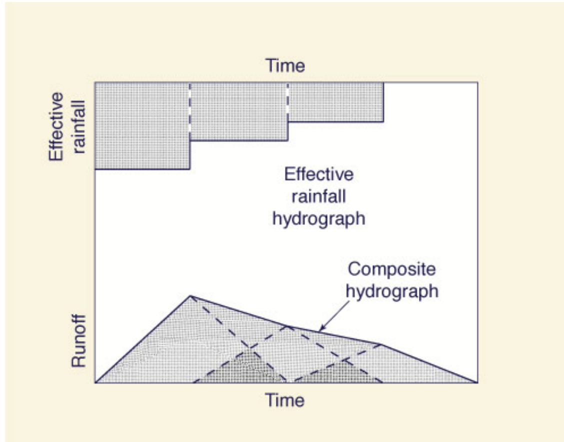

Fig. 2 (d) The unit hydrograph: Superposition.

The unit hydrograph methodology has been used in hydrologic design for nearly 100 years. It has proven to be a reliable and reproducible procedure, but the central question still remains: How to determine the unit hydrograph for a given basin? Due to the associated cost, actual measurements of unit hydrographs are very few and far between. The synthetic unit hydrographs of Snyder and NRCS stand out, but they are not without their pitfalls. The peak and shape of the actual unit hydrograph [Fig. 1 (a)] depend on the actual amount of runoff diffusion, but both the Snyder and NRCS unit hydrographs feature **specific amounts** of runoff diffusion. While Snyder's method provides a narrow range of cases, NRCS's method provides *only one* case, therefore limiting its applicability to basins similar to those used in the method's original development.

What is sorely needed is a methodology to derive a unit hydrograph whose peak and shape are not fixed by a certain amount of runoff diffusion specified by a formula, but one that varies with the actual amount, be it a small or negligible amount in the case of a steep basin, or a large or substantial amount for a mild basin. We will show here that this variability, across the broadest range possible, from zero (0) to infinity (∞) is clearly provided by the method of cascade of linear reservoirs.

4. CASCADE OF LINEAR RESERVOIRS

The methodology referred to as the *cascade of linear reservoirs* constitutes an effective way to simulate runoff diffusion in a basin. As its name implies, the method is based on the connection of several linear reservoirs in series, wherein the outflow from the first reservoir is the inflow to the second, the outflow from the second is the inflow to the third, and so on. Each reservoir in the series provides a finite amount of diffusion, with the outflow from the last reservoir reflecting the cumulative diffusion effect of all the reservoirs in the series.

Since the flow is being routed only through reservoirs, it does not provide convection, i.e., runoff concentration (the first-order process). However, the accumulated experience with the method has convincingly shown that appreciable amounts of diffusion may readily be used to model **both** convection **and** diffusion. We note that its original use, in connection with the SSARR model, used the procedure to route flow through channels, reservoirs, and basins (U.S. Army Corps of Engineers, 1975). Thus, its flexibility to model any amount of runoff diffusion, in any situation, is amply demonstrated by experience.

To derive the routing equation for the cascade of linear reservoirs, the point of start is the four-point scheme (**Ponce, 2014a: Cascade**):

$$Q_{j+1}^{\ n+1} = C_0\, Q_j^{\ n+1} + C_1\, Q_j^{\ n} + C_2\, Q_{j+1}^{\ n} \qquad (2)$$

in which Q represents discharge, whether inflow or outflow, and j and n are spatial and temporal indices respectively (Fig. 3).

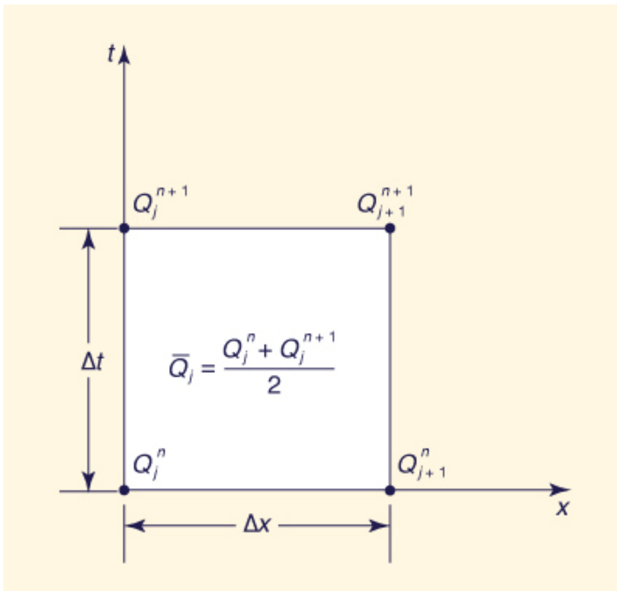

Fig. 3 Space-time discretization in the method of cascade of linear reservoirs.

The routing coefficients C_0, C_1 and C_2 are a function of the dimensionless ratio referred to as the Courant number $C = \Delta t / K$, in which K = linear reservoir storage constant. In terms of the Courant number, the routing coefficients are expressed as follows:

$$C_0 = \frac{C}{2 + C} \qquad (3a)$$

$$C_1 = C_0 \qquad (3b)$$

$$C_2 = \frac{2 - C}{2 + C} \qquad (3c)$$

For basin routing, it is convenient to define the average inflow as follows (Fig. 3):

$$\bar{Q}_j = \frac{Q_j^{\,n} + Q_j^{\,n+1}}{2} \qquad (4)$$

Substituting Eqs. 3 and Eq. 4 into Eq. 2 gives the following:

$$Q_{j+1}^{\,n+1} = 2\,C_1\,\bar{Q}_j + C_2\,Q_{j+1}^{\,n} \qquad (5)$$

Equation 5 constitutes the routing equation of the cascade of linear reservoirs. Alternatively, through some algebraic manipulation, the following equivalent form is obtained:

$$Q_{j+1}^{\,n+1} = \frac{2\,C}{2 + C}\,[\,\bar{Q}_j - Q_{j+1}^{\,n}\,] + Q_{j+1}^{\,n} \qquad (6)$$

Equation 6 is the routing equation of the SSARR model (U.S. Army Corps of Engineers, 1975). Equations 5 and 6 are in a form convenient for basin routing because the inflow is usually a rainfall hyetograph, that is, a constant average value per time interval. An example of a calculation by the cascade of linear reservoirs (Example 10-3) in given in **Ponce (2014a): Example 10-3**.

Figure 4 shows an example of the method of the cascade of linear reservoirs using the online script **ONLINEROUTING08**. The example reproduces the manual calculation shown in **Ponce (2014a): Example 10-3**, confirming the results.

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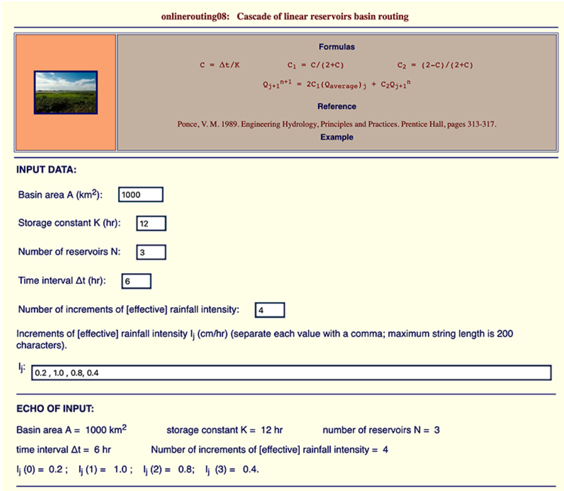


onlinerouting08: Cascade of linear reservoirs basin routing

Formulas

$C = \Delta t/K$ $C_1 = C/(2+C)$ $C_2 = (2-C)/(2+C)$

$Q_{j+1}^{n+1} = 2C_1(Q_{average})_j + C_2Q_{j+1}^n$

Reference

Ponce, V. M. 1989. Engineering Hydrology, Principles and Practices. Prentice Hall, pages 313-317.

Example

INPUT DATA:

Basin area A (km^2): 1000

Storage constant K (hr): 12

Number of reservoirs N: 3

Time interval Δt (hr): 6

Number of increments of [effective] rainfall intensity: 4

Increments of [effective] rainfall intensity I_j (cm/hr) (separate each value with a comma; maximum string length is 200 characters).

I_j: 0.2 , 1.0 , 0.8, 0.4

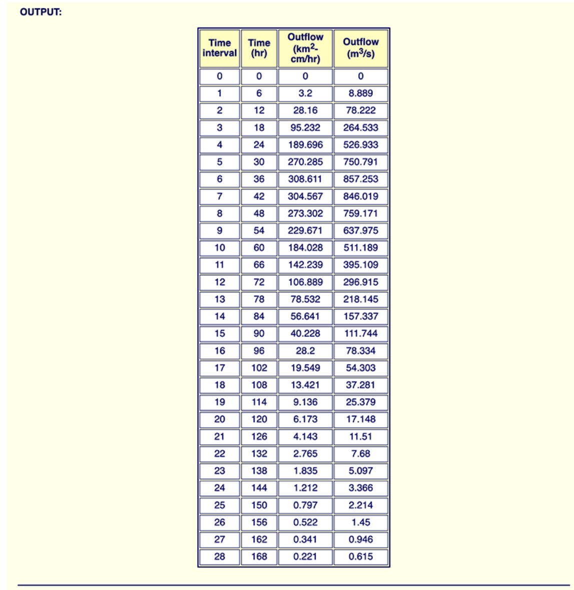
ECHO OF INPUT:

Basin area A = 1000 km^2 storage constant K = 12 hr number of reservoirs N = 3

time interval Δt = 6 hr Number of increments of [effective] rainfall intensity = 4

I_j (0) = 0.2 ; I_j (1) = 1.0 ; I_j (2) = 0.8; I_j (3) = 0.4.

Fig. 4 (a) ONLINEROUTING08: Input Data.



OUTPUT:

Time interval	Time (hr)	Outflow (km^2-cm/hr)	Outflow (m^3/s)
0	0	0	0
1	6	3.2	8.889
2	12	28.16	78.222
3	18	95.232	264.533
4	24	189.696	526.933
5	30	270.285	750.791
6	36	308.611	857.253
7	42	304.567	846.019
8	48	273.302	759.171
9	54	229.671	637.975
10	60	184.028	511.189
11	66	142.239	395.109
12	72	106.889	296.915
13	78	78.532	218.145
14	84	56.641	157.337
15	90	40.228	111.744
16	96	28.2	78.334
17	102	19.549	54.303
18	108	13.421	37.281
19	114	9.136	25.379
20	120	6.173	17.148
21	126	4.143	11.51
22	132	2.765	7.68
23	138	1.835	5.097
24	144	1.212	3.366
25	150	0.797	2.214
26	156	0.522	1.45
27	162	0.341	0.946
28	168	0.221	0.615

Fig. 4 (b) ONLINEROUTING08: Input/Output.

5. ANALYSIS OF THE CASCADE

The cascade of linear reservoirs (for short, CLR) calculates either: (a) a unit hydrograph, or (b) a flood hydrograph, given the following data: (1) basin area *A*, (2) reservoir storage constant *K*, (3) number of reservoirs *N*, (4) time interval Δ*t*, (5) number *m* of increments of effective rainfall intensity, and (6) *m* increments i_e of effective rainfall intensity. The Courant number is defined as: $C = \Delta t / K$. Effectively, the method's two parameters are *C* and *N*.

The sensitivity of the method's response to the choice of parameters is demonstrated by the example shown in **Box A** (**Ponce, 1980**). This example calculates a 6-hr unit hydrograph (1 inch of runoff) for each *C*-*N* pair of a set of five (5) Courant numbers *C* and nine (9) *N* values, for a total of forty-five (45) *C*-*N* pairs and corresponding unit hydrographs [actually, due to the great amount of diffusion, only six (6) curves can be shown in Fig. 5 (d) and four (4) in Fig. 5 (e)].

Box A. Data for Example Problem.

1. Basin area *A* = 465 square miles.
2. Reservoir storage constant *K* corresponding to the following five Courant numbers ($C = \Delta t / K$): *C* = 2.0, 0.8, 0.4, 0.2, and 0.1.
3. Nine values of *N*: 1, 2, 3, 4, 5, 6, 7, 8, and 9.
4. Time interval Δ*t* = 6 hr.
5. Number *m* of increments of effective rainfall intensity: *m* = 1 (i.e., a unit hydrograph).
6. ***One*** increment of effective rainfall intensity *i* = 1 in / 6 hr = 0.1666667 in/hr.

The results are shown in Fig. 5. The maximum possible peak flow is shown to be: Q_p = 50,000 cfs [Fig. 5 (a) and **Box B**]. Figure 5 shows that the method is able to simulate a wide range of runoff diffusion effects, from (a) zero diffusion, i.e., Q_p = 50,000 cfs, and time base (of the unit hydrograph)

T_b = 12 hr, for C = 2 and N = 1 [Fig. 5 (a)]; to (b) a substantial amount of diffusion, i.e., Q_p < 1,000 cfs for C = 0.1 and N > 4 [Fig. 5 (e)].

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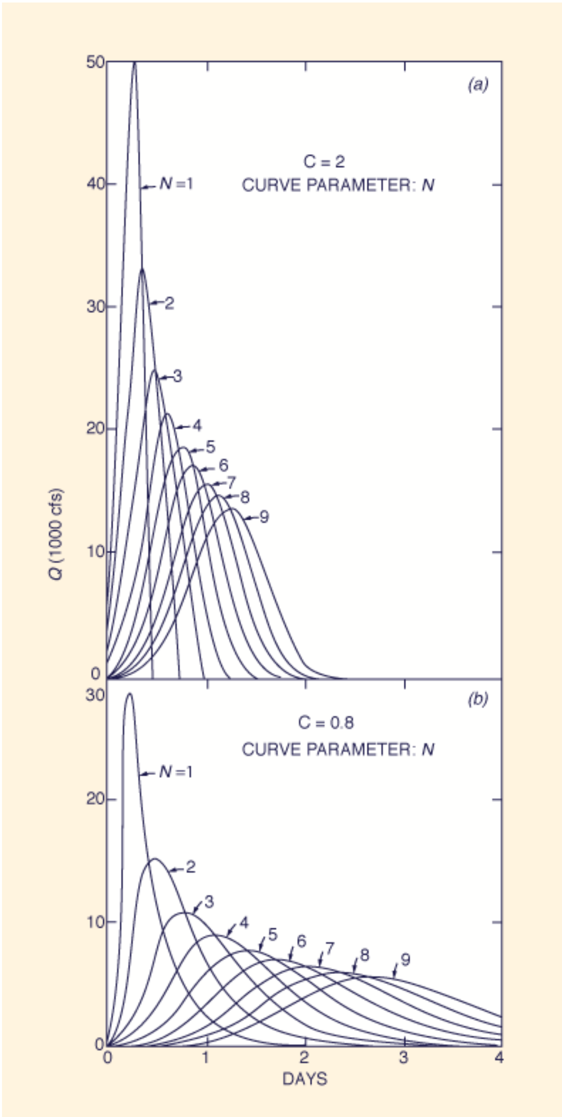

Fig. 5 (a) CLR hydrograph response:
(a) C = 2; (b) C = 0.8.

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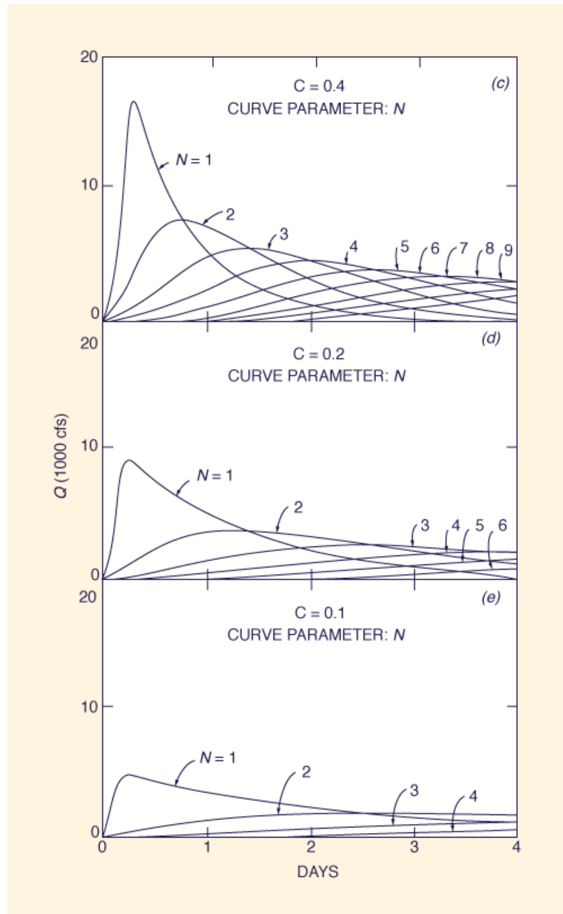

Fig. 5 (b) CLR hydrograph response:
(c) C = 0.4; (d) C = 0.2; (e) C = 0.1.

Box B. Calculation of the maximum peak flow Q_p for Example Problem.

$$Q_p = i\,A = \frac{0.1666667 \text{ in/hr} \times 465 \text{ mi}^2 \times (5{,}280 \text{ ft/mi})^2}{3{,}600 \text{ s/hr} \times 12 \text{ in/ft}} = 50{,}013 \simeq 50{,}000 \text{ cfs}$$

Significantly, the zero diffusion provided by the cascade for the case of C = 2 and N = 1 results in a triangular hydrograph, resembling that of the rational method. For any other C-N pair, diffusion **increases** with a *decrease* in Courant number and an *increase* in the number of reservoirs, as predicted by the theory. For C < 0.1 and N > 9, greater amounts of runoff diffusion than those shown in

Fig. 5 (e) are in order, confirming the theoretically infinite amount of diffusion that the method is able to provide. In fact, the diffusion will range from a value of zero (0) applicable for $C = 2$ and $N = 1$, to a value approaching infinity (∞) for C = a positive real number very close to zero, and N = a very large integer exceeding 9.

It should be pointed out that since $C = 2$ is the condition of zero diffusion, then $C > 2$ produces negative diffusion, that is, *hydrograph amplification*. In practice, diffusion requires that $\Delta t < 2\ K$, i.e., the spatial interval **must be** smaller than twice the reservoir routing constant. Therefore, $C = 2$ is effectively a practical upper limit for the Courant number.

With the capability of the method of cascade of linear reservoirs amply demonstrated for generating not only unit hydrographs (in this Section) but also composite flood hydrographs (in Section 4), all that remains is the estimation or assessment of the method's two routing parameters applicable to a given problem: (1) the Courant number C, and (2) the number of reservoirs N. This objective may be accomplished in one of the following ways: (a) by calibration, using measured field data; or (b) by estimation, using a variety of synthetic means. The following section (Section 6) elaborates on the geomorphological approach to parameter estimation.

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6. CASCADE AND GEOMORPHOLOGY

Hayami's expression for the coefficient of hydraulic diffusivity, Eq. 1, characterizes the behavior of runoff diffusion in basin/watershed free-surface flow. Equation 1 reveals that diffusion is largely controlled by the mean land surface slope. In practice, runoff diffusion manifests itself as the attenuation of the unsteady flow features, i.e., waves and other disturbances. The greater the mean land surface slope, the lesser the amount of diffusion, the latter vanishing for sufficiently large slopes. By definition, kinematic slopes, which are generally greater than 0.01 (1%), have no diffusion. Thus, mountainous terrain is essentially diffusion-free [Fig. 6 (a)] (**Ponce, 2023b**).

On the other side of the mean land surface slope range, for the extremely mild values, runoff diffusion is very substantial. As mean land surface slope attains the value of zero (i.e., a horizontal plane) diffusion becomes infinite (∞). In this case, all waves and disturburbances diffuse almost instantaneously, and the result is a pond or lake, arresting surface flow [Fig. 6 (b)].

[Click on top of figure to expand]

Fig. 6 (a) The Moyan watershed, headwaters of the La Leche river, Lambayeque, Peru (average terrain slope ≃ 0.3) (Ponce, 2008).

[Click on top of figure to expand]

Fig. 6 (b) The Upper Paraguay river near Porto Murtinho, Mato Grosso do Sul, Brazil (average terrain slope ≃ 0.000008) (Ponce, 1995).

These observations clearly point to *terrain geomorphology* as the leading factor in assessing runoff diffusion and, consequently, in determining the shape of the unit hydrograph and the related composite flood hydrograph.

Table 1 shows a tentative classification of mean land surface slope, ranging from very steep ($S_o > 0.1$), to extremely mild ($S_o < 0.00001$). The applicable cascade parameters have been estimated by **Ponce (2009b)**. The peak flow Q_{*p} and time-to-peak t_{*p} of the *generalized dimensionless unit hydrograph* (GDUH) have been calculated by **Ponce (2009a)**. Note that Q_{*p} varies from **1** (100% of its value, kinematic flow) for the *very steep* class, to **0.014** (1.4% of its value, diffusion flow) for the *extremely mild* class. Likewise, t_{*p} varies from **1** (1 time interval, kinematic flow) for the *very steep* class, to **81** (81 time intervals, diffusion flow) for the *extremely mild* class.

Table 1. Basin classification for runoff diffusion based on mean land surface slope.

Class	Mean land surface slope S_o	Cascade parameters		GDUH peak values and time-to-peak	
		C	N	Q_{*p}	t_{*p}
Very steep	> 0.1	2	1	1	1
Steep	0.01 - 0.1	1.5	2	0.472	2
Average	0.001 - 0.01	1	4	0.224	4
Mild	0.0001 - 0.001	0.5	6	0.088	11
Very mild	0.00001 - 0.0001	0.2	8	0.03	36
Extremely mild	< 0.00001	0.1	9	0.014	81

In Nature, runoff diffusion acts to obliterate the flood peaks, generally leading to milder peaks and consequently greater low flows. In extremely mild terrain geomorphology, the great amounts of diffusion lead to the occurrence of *only one flood peak per year*. This is the case of the Pantanal of Mato Grosso, in Brazil, where the extremely mild land slopes, close to S_o = 0.00001 (1 cm/km), result in only one flood peak per year in the lower basin, with its normal occurrence in May or June (Fig. 7). This fact confirms that terrain geomorphology exerts a major influence on the shape and timing of unsteady surface-water flow features.

Fig. 7 The Upper Paraguay river near Porto Murtinho, Mato Grosso do Sul, Brazil, featuring only one flood peak per year due to extreme runoff diffusion.

7. ONLINE EXAMPLES

This section shows several examples of unit hydrographs calculated by the cascade of linear reservoirs using the online scripts developed by **Ponce (2009a)**.

Example 1: ONLINE_DIMENSIONLESS_UH_CASCADE

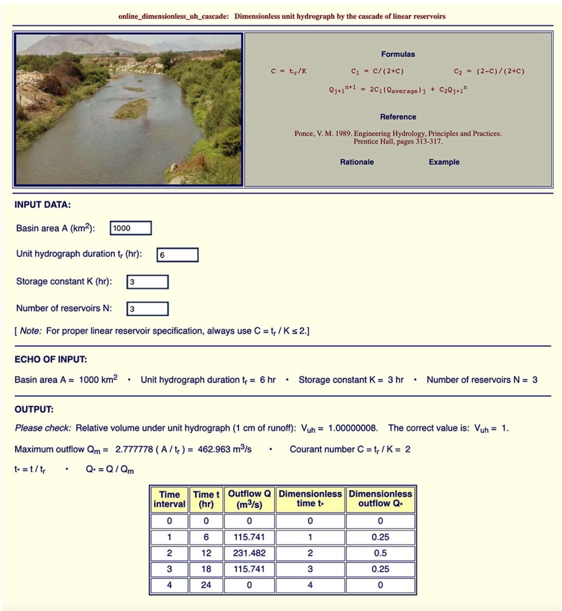
[Click on top of above link and enter the appropriate data to run online script]
[https://ponce.sdsu.edu/onlinedimensionlessuhcascade.php]

Example 1A, shown in Fig. 8 (a), calculates actual and dimensionless unit hydrographs for the following input data: (1) Basin area A = 1,000 km^2; (2) Courant number C = 2; and (3) number of reservoirs N = 3. The calculated peak outflow is Q_p = 231.482 m^3/s, occurring at t = 12 hr. The dimensionless unit hydrograph peak outflow is Q_{*p} = 0.5, occurring at t_{*p} = 2.

Example 1B, shown in Fig. 8 (b), uses the same basin area A and Courant number C as Example 1A, but increases the number of reservoirs to N = 5. The calculated peak outflow is Q_p = 173.611 m^3/s, occurring at t = 18 hr. The dimensionless unit hydrograph peak outflow is Q_{*p} = 0.375, occurring at t_{*p} = 3.

Examples 1A and 1B show that increasing the number of reservoirs from 3 (Example 1A) to 5 (Example 1B) has decreased (i.e., attenuated, diffused) the unit hydrograph peak outflow Q_p from 231.482 to 173.611 m^3/s, while increasing the time-to-peak t_p from 12 to 18 hr. The dimensionless unit hydrograph peak outflow Q_{*p} has decreased from 0.5 to 0.375, while the dimensionless time-to-peak t_{*p} has increased from 2 to 3.

[Click on top of figure to expand]


online_dimensionless_uh_cascade: Dimensionless unit hydrograph by the cascade of linear reservoirs

Formulas

C = t_r/K   C_1 = C/(2+C)   C_2 = (2-C)/(2+C)

$Q_{j+1}^{n+1} = 2C_1 (Q_{average})_j + C_2 Q_{j+1}^n$

Reference

Ponce, V. M. 1989. Engineering Hydrology, Principles and Practices. Prentice Hall, pages 313-317.

Rationale   Example

INPUT DATA:

Basin area A (km^2): 1000

Unit hydrograph duration t_r (hr): 6

Storage constant K (hr): 3

Number of reservoirs N: 3

[*Note:* For proper linear reservoir specification, always use $C = t_r / K \le 2$.]

ECHO OF INPUT:

Basin area A = 1000 km^2 • Unit hydrograph duration t_r = 6 hr • Storage constant K = 3 hr • Number of reservoirs N = 3

OUTPUT:

Please check: Relative volume under unit hydrograph (1 cm of runoff): V_{uh} = 1.00000008. The correct value is: V_{uh} = 1.

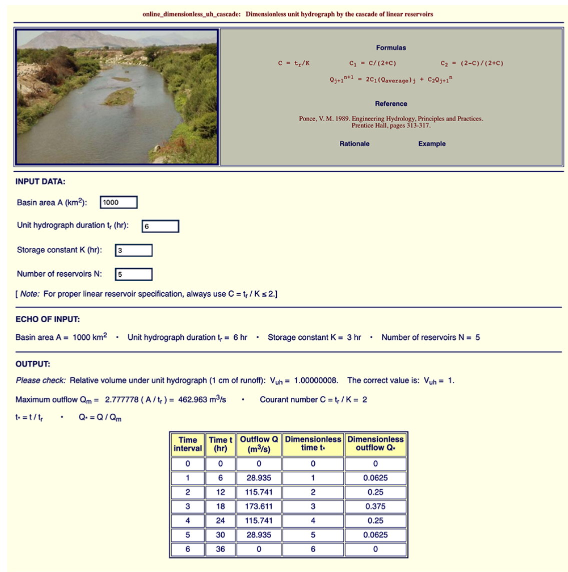
Maximum outflow Q_m = 2.777778 (A / t_r) = 462.963 m^3/s • Courant number $C = t_r / K$ = 2

$t_* = t / t_r$ • $Q_* = Q / Q_m$

Time interval	Time t (hr)	Outflow Q (m^3/s)	Dimensionless time t_*	Dimensionless outflow Q_*
0	0	0	0	0
1	6	115.741	1	0.25
2	12	231.482	2	0.5
3	18	115.741	3	0.25
4	24	0	4	0

Fig. 8 (a) Example 1A: Unit hydrograph for C = 2 and N = 3.

[Click on top of figure to expand]


online_dimensionless_uh_cascade: Dimensionless unit hydrograph by the cascade of linear reservoirs

Formulas

C = t_r/K   C_1 = C/(2+C)   C_2 = (2-C)/(2+C)

$Q_{j+1}^{n+1} = 2C_1 (Q_{average})_j + C_2 Q_{j+1}^n$

Reference

Ponce, V. M. 1989. Engineering Hydrology, Principles and Practices. Prentice Hall, pages 313-317.

Rationale   Example

INPUT DATA:

Basin area A (km^2): 1000

Unit hydrograph duration t_r (hr): 6

Storage constant K (hr): 3

Number of reservoirs N: 5

[*Note:* For proper linear reservoir specification, always use $C = t_r / K \le 2$.]

ECHO OF INPUT:

Basin area A = 1000 km^2 • Unit hydrograph duration t_r = 6 hr • Storage constant K = 3 hr • Number of reservoirs N = 5

OUTPUT:

Please check: Relative volume under unit hydrograph (1 cm of runoff): V_{uh} = 1.00000008. The correct value is: V_{uh} = 1.

Maximum outflow Q_m = 2.777778 (A / t_r) = 462.963 m^3/s • Courant number $C = t_r / K$ = 2

$t_* = t / t_r$ • $Q_* = Q / Q_m$

Time interval	Time t (hr)	Outflow Q (m^3/s)	Dimensionless time t_*	Dimensionless outflow Q_*
0	0	0	0	0
1	6	28.935	1	0.0625
2	12	115.741	2	0.25
3	18	173.611	3	0.375
4	24	115.741	4	0.25
5	30	28.935	5	0.0625
6	36	0	6	0

Fig. 8 (b) Example 1B: Unit hydrograph for C = 2 and N = 5.

Example 2: ONLINE_SERIES_UH_CASCADE

[Click on top of above link and enter C = 1 to run online script]
[https://ponce.sdsu.edu/onlineseriesuhcascade.php]

This example shows the calculation of dimensionless unit hydrographs for a given Courant number C and the number of reservoirs in the range $1 \le N \le 10$. For this example, C = 1. The results are shown in Fig. 9. It is seen that diffusion increases with the number of reservoirs. For N = 1, Q_{*p} = 0.667 at t_{*p} = 1; for N = 10, Q_{*p} = 0.132 at t_{*p} = 10.

[Click on top of figure to expand]

online_series_uh_cascade: Series of dimensionless CLR unit hydrographs

Formulas

$C_1 = C/(2+C)$ $C_2 = (2-C)/(2+C)$

$Q_{j+1}^{n+1} = 2C_1(Q_{average})_j + C_2Q_{j+1}^n$

Reference

Ponce, V. M. 1989. Engineering Hydrology, Principles and Practices. Prentice Hall, pages 313-317.

Rationale

INPUT DATA:

Courant number C: 1

[Recommended range 0.1 ≤ C ≤ 2.] [To test program, enter 1 in input box and click 'Calculate']

OUTPUT:

Dimensionless time t*	Dimensionless discharge Q* for indicated number of linear reservoirs N									
	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0.667	0.222	0.074	0.025	0.008	0.003	0.001	0	0	0
2	0.222	0.37	0.222	0.107	0.047	0.019	0.008	0.003	0.001	0
3	0.074	0.222	0.272	0.2	0.118	0.061	0.029	0.013	0.006	0.002
4	0.025	0.107	0.2	0.224	0.181	0.12	0.07	0.038	0.019	0.009
5	0.008	0.047	0.118	0.181	0.195	0.165	0.119	0.075	0.044	0.024
6	0.003	0.019	0.061	0.12	0.165	0.175	0.153	0.116	0.078	0.049
7	0.001	0.008	0.029	0.07	0.119	0.153	0.16	0.143	0.112	0.08
8	0	0.003	0.013	0.038	0.075	0.116	0.143	0.149	0.135	0.109
9	0	0.001	0.006	0.019	0.044	0.078	0.112	0.135	0.139	0.128
10	0	0	0.002	0.009	0.024	0.049	0.08	0.109	0.128	0.132
11	0	0	0.001	0.004	0.012	0.028	0.052	0.08	0.106	0.122
12	0	0	0	0.002	0.006	0.016	0.032	0.055	0.08	0.103
13	0	0	0	0.001	0.003	0.008	0.019	0.035	0.057	0.08
14	0	0	0	0	0.001	0.004	0.01	0.021	0.038	0.058
15	0	0	0	0	0.001	0.002	0.006	0.012	0.024	0.04
16	0	0	0	0	0	0.001	0.003	0.007	0.014	0.026
17	0	0	0	0	0	0	0.001	0.004	0.008	0.016
18	0	0	0	0	0	0	0.001	0.002	0.005	0.01
19	0	0	0	0	0	0	0	0.001	0.003	0.006
20	0	0	0	0	0	0	0	0	0.001	0.003
21	0	0	0	0	0	0	0	0	0.001	0.002
22	0	0	0	0	0	0	0	0	0	0.001
23	0	0	0	0	0	0	0	0	0	0

Fig. 9 Example 2: Dimensionless unit hydrographs for C = 1 and 1 ≤ N ≤ 10.

Example 3: ONLINE_ALL_SERIES_UH_CASCADE

[Click on top of above link and then Click "Calculate" to run online script in default mode for the entire specified series]
[https://ponce.sdsu.edu/onlineallseriesuhcascade.php]

This example shows the calculation of dimensionless unit hydrographs for six (6) Courant numbers in the range 0.1 ≤ C ≤ 2.0 and ten (10) cascades of reservoirs (N = number of reservoirs in each cascade), with 1 ≤ N ≤ 10. The script produces sixty (6 × 10 = 60) dimensionless unit hydrographs, encompassing all combinations of Courant numbers C = 0.1, 0.2, 0.5, 1.0, 1.5, and 2.0, and number of reservoirs N = 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. As predicted by the theory, it is confirmed that runoff diffusion *increases* with a decrease in Courant number C and an increase in the number N of reservoirs in the cascade.

Example 4: PRACTICAL APPLICATION

[https://ponce.sdsu.edu/onlinedimensionlessuhcascade.php]

Problem statement. Calculate the t_r = 2-hr unit hydrograph for a basin with surface area A = 300 km^2 and mean land surface slope S_o = 0.005.

Answer. According to Table 1, this basin is classified as *mild*. Therefore, the applicable cascade parameters are estimated from this table as C = 0.5 and N = 6. The time interval $\Delta t \equiv t_r$ = 2 hr, and the reservoir storage constant is: $K = \Delta t / C$ = 2 / 0.5 = 4 hr.

We run **ONLINE_DIMENSIONLESS_UH_CASCADE** with A = 300 km^2, unit hydrograph duration t_r = 2 hr, storage constant K = 4 hr, and number of reservoirs N = 6. **Box C** shows the calculation of the maximum peak outflow, verified to be Q_p = 416.6667 m^3/s. The volume under the unit hydrograph is verified to be 1 cm of runoff (see Fig. 10).

The resulting dimensional and dimensionless unit hydrographs are shown in Fig. 10. The dimensional unit hydrograph peak outflow is Q_p = 36.495 m^3/s at the time-to-peak t_p = 22 hr. The dimensionless unit hydrograph peak outflow is Q_{*p} = 0.0876 at the dimensionless time-to-peak t_{*p} = 11.

Box C. Calculation of the maximum peak flow Q_p for Example 4.

$$Q_p = i\,A = \frac{0.5 \text{ cm/hr} \times 300 \text{ km}^2 \times (1{,}000 \text{ m/km})^2}{3{,}600 \text{ s/hr} \times 100 \text{ cm/m}} = 416.6667 \text{ m}^3\text{/s}$$

[Click on top of figure to expand]

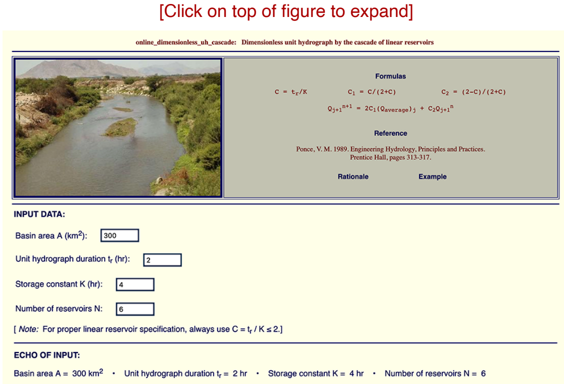
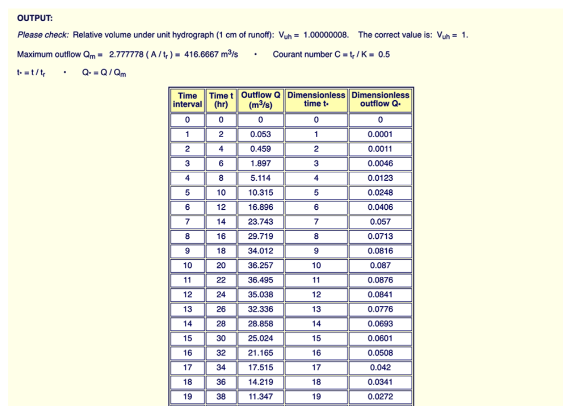


Fig. 10 (a) Example 4: Input Data.

[Click on top of figure to expand]



OUTPUT:

Please check: Relative volume under unit hydrograph (1 cm of runoff): V_{uh} = 1.00000008. The correct value is: V_{uh} = 1.

Maximum outflow Q_m = 2.777778 (A / t_r) = 416.6667 m^3/s • Courant number $C = t_r / K$ = 0.5

$t_* = t / t_r$ • $Q_* = Q / Q_m$

Time interval	Time t (hr)	Outflow Q (m^3/s)	Dimensionless time t*	Dimensionless outflow Q*
0	0	0	0	0
1	2	0.053	1	0.0001
2	4	0.459	2	0.0011
3	6	1.897	3	0.0046
4	8	5.114	4	0.0123
5	10	10.315	5	0.0248
6	12	16.896	6	0.0406
7	14	23.743	7	0.057
8	16	29.719	8	0.0713
9	18	34.012	9	0.0816
10	20	36.257	10	0.087
11	22	36.495	11	0.0876
12	24	35.038	12	0.0841
13	26	32.336	13	0.0776
14	28	28.858	14	0.0693
15	30	25.024	15	0.0601
16	32	21.165	16	0.0508
17	34	17.515	17	0.042
18	36	14.219	18	0.0341
19	38	11.347	19	0.0272

Fig. 10 (b) Example 4: Output.

8. SUMMARY

A review of the method of cascade of linear reservoirs for unit hydrograph development is presented, explained, and clarified. Section 1 compares the cascade with the synthetic unit hydrographs of Snyder and NRCS. Section 2 presents basin modeling concepts, including Hayami's diffusivity, which characterizes the diffusion of one-dimensional free-surface flows. Section 3 reviews concepts of unit hydrograph theory. Section 4 explains the methodology of cascade of linear reservoirs, deriving the routing equation and displaying actual online example calculations. Section 5 provides an analysis of the cascade, focusing on its capability to model a broad range of runoff diffusion effects, from zero (0) to infinity (∞). Section 6 describes the geomorphological approach to estimate the parameters of the cascade. Section 7 provides several examples of online calculation, including an actual practical application (Example 4).

The cascade of linear reservoirs is predicated on its capability to model a broad range of runoff diffusion effects, enabling increased accuracy for simulating synthetic unit hydrographs and associated flood hydrographs. The online computational capability enhances the method's utility for the effective modeling of runoff diffusion to solve a wide variety of flooding problems.

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